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Thin Watts-Strogatz networks

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A modified version of the Watts-Strogatz (WS) network model is proposed, in which the number of shortcuts scales with the network size N as N^{α}, with $\alpha<1$. In these networks, the ratio of the number of shortcuts to the network size approaches zero as $N\to\infty$, whereas in the original WS model, this ratio is constant. We call such networks "thin Watts-Strogatz networks." We show that even though the fraction of shortcuts becomes vanishingly small for large networks, they still cause a kind of small-world effect, in the sense that the length L of the network increases sublinearly with the size. We develop a mean-field theory for these networks, which predicts that the length scales as $N^{1-\alpha}\ln N$ for large N. We also study how a search using only local information works in thin WS networks. We find that the search performance is enhanced compared to the regular network, and we predict that the search time τ scales as $N^{1-\alpha/2}$. These theoretical results are tested using numerical simulations. We comment on the possible relevance of thin WS networks for the design of high-performance low-cost communication networks.

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The mathematical model for a small-world network proposed in the seminal paper by Watts and Strogatz [1,2] sparked a widespread interest on the subject of complex networks (for reviews, see Refs. [3–5]). Their model, known as the Watts-Strogatz (WS) model, consists of a network in which the nodes are arranged on a ring, every node being initially connected to its $2k$ nearest neighbors (k on each side). The WS network is generated by letting each link have a probability p of being rewired randomly, so that it may connect to any other node on the ring with uniform probability. This means that besides the local connections, there are a number n_{sc} of *shortcuts*, that is, links connecting pairs of randomly chosen nodes on the ring. The shortcuts are nonlocal, long-range connections. An important parameter in this model is the ratio $\beta=n_{sc}/N$ of the number of shortcuts to the total number of nodes N. The shortcut links represent a kind of topological disorder that exists alongside the regularity of the local connections. The case $\beta=0$ is the regular ring network, and $\beta\to k$ is the limit of a very disordered network. Intermediate values of β correspond to networks in between these two extremes, partly regular and partly disordered. A crucial quantity is the *length* L, the average shortest path connecting two nodes in the network. In other words, L measures how many hops one has to do, on average, to move from one node to another. It can be shown that, for any constant nonzero β, L increases with the number of nodes N as [6]

$$L \sim \ln N. \tag{1}$$

This slow increase of the length with the network size means that even in very large networks, it takes typically only a few hops to connect any two nodes; this is what is commonly referred to as the so-called small-world phenomenon. This happens even if β is very small (but nonzero), so that even a very small number of shortcuts (compared to the total number of links) is enough to cause the small-world effect, as long as β does not vanish as N increases.

The result (1) holds if the shortcut fraction β is a constant greater than zero. If we consider a family of WS networks with increasing size N, a constant β corresponds to the shortcut number n_{sc} increasing linearly with N, $n_{sc}\sim N$. But what happens if the number of shortcuts increases more slowly than the network size? A simple instance of this is n_{sc} scaling as a power of N,

$$n_{sc} \sim N^{\alpha}. \tag{2}$$

For $0<\alpha<1$, even though the number of shortcuts goes to infinity in the "thermodynamic limit" $N\to\infty$, the fraction of shortcuts scales as

$$\beta \sim n_{sc}/N \sim N^{\alpha-1}, \tag{3}$$

and therefore $\beta\to 0$ for $N\to\infty$. This means that the shortcuts become infinitely sparse in these networks, as their size grows. We call these networks "thin Watts-Strogatz networks," or "thin WS networks" for short. The goal of this paper is to study the statistical properties of these networks, and in particular to study how the length depends on the size, for a given value of the parameter α. One could think that, since the fraction of the shortcuts approaches zero for large N, the length would scale linearly with N, as is the case for regular networks. By using a mean-field approach, however, we show that L scales in fact as $L\sim N^{1-\alpha}\ln N$. Thus, the length increases more slowly than linearly, although not as slowly as the pure logarithmic growth of the original WS model. This means that the presence of the shortcuts has dramatic effects on the network connectivity, even though their fraction becomes vanishingly small for large N. We also investigate the dynamics of a local search on thin WS networks. We find that the search time decreases with N faster than for the regular network, showing that the vanishing fraction of shortcuts also has an important effect in the dynamical processes that take place on these networks.

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We first very briefly review the version of the Watts-Strogatz network we use in this paper [6,7]. The skeleton of the WS network is a regular network in which the nodes are arranged on a ring, with the neighboring nodes being connected. In the general model, the nodes are connected to its $2k$ nearest neighbors, but we shall consider throughout this paper only the simplest case in which there are links only to the nearest neighbors ($k=1$); the results we obtain are independent of the precise value of k. Besides these local connections, there are also random long-range connections, called *shortcuts*. These are links connecting randomly chosen pairs of nodes, which we add to the ring, as opposed to rewiring, as in the original WS model [7]. We denote the number of such shortcuts by n_{sc}, and the ratio n_{sc}/N of the number of shortcuts to the number of nodes by β. For a constant nonzero β, it is known that the mean average path length L increases logarithmically with N. Using a mean-field model, Newman *et al.* [6] found an analytical expression for L,

$$L \approx \frac{1}{\beta}\ln(\beta N). \tag{4}$$

This approximation is valid for large N and for a large number of shortcuts, $n_{sc}=\beta N \gg 1$.

We first consider a theory for the scaling of the length in thin WS networks. Assuming that the number of shortcuts n_{sc} scales as a power law with the number of nodes N, as in Eq. (2), the scaling of the fraction of shortcuts β is given by Eq. (3). Therefore, $\beta \to 0$ for large N if $\alpha < 1$. In the limit of large network sizes, the fraction of shortcuts vanishes, even though their number still diverges, as seen from Eq. (2).

To obtain a mean-field model for the case where β is not constant, but instead depends on the network size N according to Eq. (3), we just have to realize that Eq. (4) is still valid even with a nonconstant β, as long as the total number of shortcuts n_{sc} diverges as $N \to \infty$. From Eq. (2), we see that this condition is fulfilled for any $\alpha > 0$. Substituting Eq. (3) in Eq. (4), we find the following scaling law for the length:

$$L \sim N^{1-\alpha}\ln N. \tag{5}$$

This expression predicts a power-law behavior for L with a multiplicative logarithmic correction for thin WS networks ($\alpha < 1$), and a pure logarithmic behavior for the usual case β=const, as it should. In order to test this prediction, we simulate WS networks with different sizes, with a number of shortcuts between randomly chosen nodes following Eq. (2). The length L is computed and plotted as a function of N. The result is shown in Fig. 1, for three different values of α. The result of nonlinear fitting of the data to Eq. (5) is also displayed in the figure, and it shows a good agreement with the predicted values of α, albeit with small deviations, especially for small α. These small but consistent deviations may be due to finite-size effects.

In a completely regular network, with no shortcuts, the length increases linearly with size, $L \sim N$. The result of the mean-field theory, given by Eq. (5) and tested in Fig. 1, shows that L increases more slowly with N in thin WS networks than in regular ones, *even though the fraction β of long-range links goes to zero for large sizes*: $\beta \to 0$ for $N \to \infty$, if $\alpha < 1$. In other words, despite the fact that in the thermodynamic limit of large N the fraction of shortcuts is negligible, their presence has major consequences for the topology of the network, giving rise to a kind of small-world effect, in which the length scales as a power law of the network size, as opposed to the logarithmic scaling found in usual small worlds with constant β. This means that in thin WS networks, the length increases more rapidly with size than in the usual WS networks, and thus the small-world effect is less pronounced. It is nevertheless remarkable that any small-world effect occurs at all for a vanishing fraction of shortcuts. If each link has a cost attached to it, Eq. (5) implies that one can build a network with almost the same cost of a regular network (in the limit of large network sizes), but with a much better connectivity (measured by the length L). This could be important in the design of communication networks, for example, especially if shortcuts are more expensive than local connections, which is a good assumption in networks like the Internet.

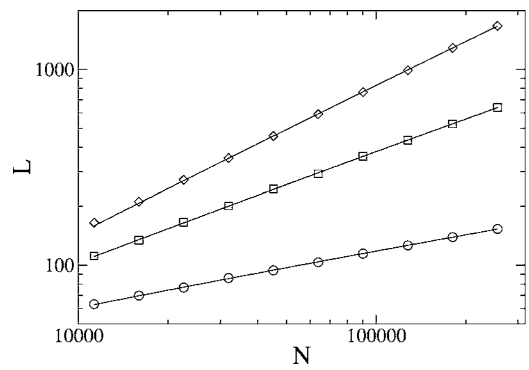

FIG. 1. Length L as a function of the number N of nodes of Watts-Strogatz networks. Diamonds correspond to α=0.3, squares to α=0.5, and circles to α=0.8. Each data point is the average of the length over 20 realizations of the networks. The continuous lines are the result of nonlinear fitting of the data to the function $L=\text{const}\times N^{\gamma}\ln N$. The fitting gives γ=0.66 for α=0.3 (compared to a predicted value of 0.7); γ=0.47 for α=0.5 (predicted value: 0.5); and γ=0.19 for α=0.8 (predicted value: 0.2). The predicted values, from Eq. (5), are given by $\gamma=1-\alpha$.

We have also investigated the dynamics of local search in thin WS networks. In a local search, one starts from some node in the network, and by following the links tries to find some target node. It is assumed that the topology of the network is not perfectly known, and that the searcher can only see its immediate neighborhood. Previous works have shown [8–15] that the average time it takes to reach the target node, the search time τ, does not in general follow the same scaling as the length L. The reason is that the lack of complete information about the network topology results in nonoptimal paths from the beginning node to the target. In Ref. [14], this problem was studied in the usual Watts-Strogatz small-world network (constant β). The search procedure used there was as follows [14]. One starts at a randomly chosen node, and wants to reach a target node (also picked at random). One of the neighbors of the current node is chosen to hop to. The chosen node is the one that appears

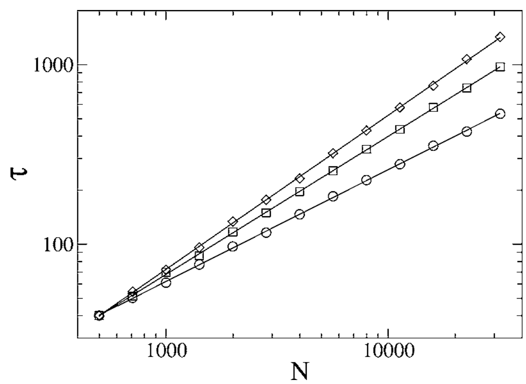

FIG. 2. Search time τ as a function of the network size N, for thin WS networks. The diamonds are the results of numerical simulation with $\alpha=0.3$, the squares are data for $\alpha=0.5$, and the circles are data for $\alpha=0.8$. Each point is obtained from the average of the search time over 20 networks with the shortcuts randomly chosen, corresponding to the same α and N, and the search time of each individual network is the average over 200 searches, with initial and target nodes chosen randomly. The continuous line is the result of fitting the numerical data to a power law $\tau \sim N^{\gamma}$. The values of γ found by fitting are $\gamma=0.86$ for $\alpha=0.3$ (the predicted value is 0.85); $\gamma=0.77$ for $\alpha=0.5$ (the predicted value is 0.75); and $\gamma=0.62$ for $\alpha=0.8$ (the predicted value is 0.6). From Eq. (7), the predicted values are $\gamma=1-\alpha/2$.

to be closest to the target node, according to a metric defined on the network, which embodies an imperfect knowledge about the global network structure. This metric gives a clue as to how near or how far we are from the target, without being a perfect guide. In Watts-Strogatz networks, the natural metric is just the separation along the ring. One thus hops to the neighboring node that is closest to the target in the sense that it takes the smallest number of hops *along the neighbors on the ring* to get there. The process is repeated until the target is reached. Using a mean-field approximation to this dynamics, it was found [14] that the search time τ scales with N and β as [14]

$$\tau \sim \left(\frac{N}{\beta}\right)^{1/2}. \tag{6}$$

This result is valid for a large network size ($N \gg 1$), a small shortcut fraction ($\beta \ll 1$), and a large number of shortcuts ($N\beta \gg 1$) [14]. All these conditions are satisfied for thin WS networks of large size, as Eqs. (2) and (3) show. Therefore, the above expression is also valid for thin WS networks, with β given by Eq. (3). Making this substitution, we get

$$\tau \sim N^{1-\alpha/2}. \tag{7}$$

For $\alpha=1$ we recover the $N^{1/2}$ scaling of usual WS networks. For thin WS networks ($\alpha<1$), τ increases faster with N than for usual small worlds, as expected. However, τ still increases more slowly than linearly, as long as $\alpha>0$. This means that even though as $N\to\infty$ the shortcut fraction vanishes, this vanishing fraction causes a considerable decrease in the search time, making it grow sublinearly with N. We have simulated this search process in thin WS networks of different sizes, according to the search algorithm described in detail in Ref. 14. The results are plotted in Fig. 2, and the scaling of τ with N found by fitting agrees well with the prediction of Eq. (7).

This result shows that not only static topological features have nontrivial behavior in thin WS networks: dynamical processes taking place on the network are also substantially affected by the vanishing fraction of shortcut links. Again, this might have important consequences for the design of communication networks and other artificial networks, because it implies that it is possible to build low-cost networks with a small number of long-range links which have a much better performance than an equivalent totally regular network.

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